Solutions to Selected Exercises

A Note to the Reader

I provide a solution here to any exercise that falls into one of two categories. (1) It's typical of the exercises in its set, and grasping the details of the solution that I provide should help you with most of the other exercises in the set. (2) The exercise is a bit too difficult unless you're *exceedingly* ambitious. I alert you to such exercises by identifying them in red font as *difficult, very difficult,* or *ridiculously difficult*. In spite of their difficulty, I hope that you *are* exceedingly ambitious, and therefore I encourage you to try each of them *before* consulting the solution that I provide.

I also note exercises that are too difficult to prove at the point where they appear in the text—but not too difficult to prove subsequently, i.e., after you've proved a later exercise that I specify. I encourage you to return to, and to work your way through, on your own, the given exercise—the exercise that initially is too difficult—after you've proved the later exercise.

The following are the exercises for which I provide solutions or which are too difficult to attempt without having first proved later exercises. I write 'Postpone and Return' next to the latter. In some cases, I offer readily provable variations of the exercises that are too difficult, and I note each of them as such in red font.

Chapter One

Pp. 39-40: 5, 10
P. 46: 5, 10, 15

Chapter Two

Pp. 58-59: 5, 10, 15
Pp. 66-67: 5
Pp. 73-74: 5, 9, 10
P. 78: 5, 10
P. 97: the fifth and tenth columns.
P. 101: I.5, II.5

Chapter Three

P. 154: 5, 10
P. 165: 5, 10, 15
P. 168: 5
Pp. 173-174: 3 (but I add two readily provable variations), 5, 6 (Postpone and Return—but I add a readily provable variation), 8, 10, 13, 15 (Postpone and Return—but I add a readily provable variation), 16 (Postpone and Return—but I add a readily provable variation).
Pp. 196-7: 5, 9, 15.

Chapter Four

P. 204: 5, 6, 10
P. 217: 5, 10
Pp. 238-239: 4, 5, 10
Pp. 241-242: 1.5, 1.10 (first half only); II.5, II.10, II.12, II.15

Chapter Five

Pp. 273-274: 5, 10, 15, 20, 25, 30, 35, 40
P. 288: 5, 10
Pp. 293-294: 5, 10, 15, 20, 25, 30, 34

Chapter Six

Pp. 304-305: 5, 10, 15, 20, 25, 30, 31, 35
Pp. 313-314: 5, 7, 10
P. 317: 5, 10
Pp. 322-324: Section I: 5, 10, 12, 15; Section II: 5, 10, 15, 20, 25
Pp. 344-345: Section I: 2, 5, 7, 10; Section II: 2, 5, 7, 10
P. 360: II.5, III.5, IV.5

Chapter Seven

Pp. 379-380: 5, 10, 15, 20
P. 414: S182, S187

Chapter Eight

Pp. 427-428: Section 1: S195, S196; Section 3: S206, S207 (without QN); Section 4: S211, S216
Pp. 440-442: S231, S234, S237, S239, S244
P. 447: T42, T47

Chapter One

Pp. 39-40

Please note. The instructions for the exercises on pp. 39-40 should read as follows: “Determine the truth-values of the following propositions if (1) you assign T to ‘P’, F to ‘Q’, T to ‘R’, and F to ‘S’; and (2) you assign F to ‘P’, T to ‘Q’, F to ‘R’, and T to ‘S’. Show all your work. Place a truth-value under each atomic and under each connective.”

5.(1) —(—P ↔ —R)
F FT T FT
5.(2) —(—P ↔ —R)
F TF T TF

10.(1) [P → (Q ↔ R)] → —{(—P ∧ —S) ∨ —[(P ∨ —R) → Q]}
T F F F T T F FT F TF T T T T FT F F

10.(2) [P → (Q ↔ R)] → —{(—P ∧ —S) ∨ —[(P ∨ —R) → Q]}
F T T F F T T TF F FT F F F T T F T T

P.46

5. If Ardbeg fails to howl if Bobo fails to howl, then Coco fails to howl.
(—B → —A) → —C

10. If both Ardbeg and Bobo howl if and only if Coco doesn’t howl, and if Dagbar growls if and only if Egvalt doesn’t growl, then either Ardbeg howls and Coco doesn’t howl, or Dagbar growls and Egvalt doesn’t growl.
{[(A ∧ B) ↔ —C] ∧ (D ↔ —E)}→ [(A ∧ —C) ∨ (D ∧ —E)]

15. It’s untrue that Ardbeg fails to howl if and only if Egvalt fails to growl.
—(—A ↔ —E)

Chapter Two

Pp. 58-59

5. Ardbeg doesn't howl unless it so happens both that Bobo howls and that Coco fails to howl.
——A → (B ∧ —C)
or: —(B ∧ —C) → —A
or: —A ∨ (B ∧ —C)

10. Ardbeg and Bobo do not both howl unless either Dagbar or Egvalt fails to growl.
——(A ∧ B) → (—D ∨ —E)
or: —(—D ∨ —E) → —(A ∧ B)
or: —(A ∧ B) ∨ (—D ∨ —E)

15. Referring to Dagbar and Egvalt: at most one of them growls.
—(D ∧ E)

Pp. 66-67

5.

P	Q	R	P	↔	(Q	↔	R)	Q	↔	(P	↔	R)	R	↔	(P	↔	Q)
T	T	T	T	**T**	T	T	T	T	**T**	T	T	T	T	**T**	T	T	T
T	T	F	T	**F**	T	F	F	T	**F**	T	F	F	F	**F**	T	T	T
T	F	T	T	**F**	F	F	T	F	**F**	T	T	T	T	**F**	T	F	F
T	F	F	T	**T**	F	T	F	F	**T**	T	F	F	F	**T**	T	F	F
F	T	T	F	**F**	T	T	T	T	**F**	F	F	T	T	**F**	F	F	T
F	T	F	F	**T**	T	F	F	T	**T**	F	T	F	F	**T**	F	F	T
F	F	T	F	**T**	F	F	T	F	**T**	F	F	T	T	**T**	F	T	F
F	F	F	F	**F**	F	T	F	F	**F**	F	T	F	F	**F**	F	T	F

Valid

Note that the three sentences—the two premises and the conclusion—are logically equivalent: they come out true under the same interpretations (the first, fourth, sixth, and seventh), and false under the same interpretations (the second, third, fifth, and eighth).

<u>Pp. 73-74</u>

5. (P ∨ —Q) → (P ∧ —R)
F F FT T F F TF
(—P ∧ —R) → (— —P ∨ —Q)
TF T T F F F TF F FT

Invalid

Counterexample:

P	Q	R
F	T	F

9. P ∧ (Q → R)
T T T T T
—[(—Q ∧ R) ∨ —P]
T FT F T F FT
—(—P ∧ Q)
F TF T T

Valid

10. —(P ∧ —Q) ↔ (R ∨ —P)
T F F FT T T T TF
—(Q ∨ R) ↔ —R
F T T T T FT
Q → P
T F F

Invalid

Counterexample:

P	Q	R
F	T	T

<u>P. 78</u>

5. —(P → Q) ∧ —(P ∧ —Q)
T T F F T T T F FT

or: —(P → Q) ∧ —(P ∧ —Q)
T T F F T T F F TF

A semantic contradiction: there's no interpretation that makes the *wff* true.

10. (—P → —Q) ↔ (—Q → —P)
FT T FT T FT T FT
FT T TF F TF F FT

A contingent *wff:* there's at least one interpretation that makes the *wff* true and there's at least one interpretation that makes it false.

P. 97

The fifth column defines '→' for 'Q → P' (as distinct from 'P → Q').
The tenth column defines '—(P ↔ Q)'.

P. 101

I. 5. P ↔ Q
(P → Q) ∧ (Q → P)
—(P ∧ —Q) ∧ —(Q ∧ —P)
(P | —Q) ∧ (Q | —P)
[P | (Q | Q)] ∧ [Q | (P | P)]
{[P | (Q | Q)] | [Q | (P | P)]} | {[P | (Q | Q)] | [Q | (P | P)]}

II. 5. [P | (P | Q)] | (P | R)
[P | (P | Q)] | —(P ∧ R)
[P | —(P ∧ Q)] | —(P ∧ R)
—[P ∧ —(P ∧ Q)] | —(P ∧ R)
— {—[P ∧ —(P ∧ Q)] ∧ —(P ∧ R)}
— —[P ∧ —(P ∧ Q)] ∨ — —(P ∧ R)
[P ∧ —(P ∧ Q)] ∨ — —(P ∧ R)
[P ∧ —(P ∧ Q)] ∨ (P ∧ R)
[P ∧ (—P ∨ —Q)] ∨ (P ∧ R)
(P ∧ —P) ∨ (P ∧ —Q) ∨ (P ∧ R)
(P ∧ —Q) ∨ (P ∧ R)
P ∧ (—Q ∨ R)
P ∧ (Q → R)

Is it clear why it's legitimate to eliminate '(P ∧ —P)' in the move from the fourth-to-last line to the third-to-last line? It's because a string of disjuncts, one of whose disjuncts is a semantic contradiction, e.g., '(P ∧ —P)', is logically equivalent to the string of disjuncts without the contradiction. (Intuitively: '(△ ∧ —△) ∨ ☐' is logically equivalent to '☐': if '☐' is true, the entire disjunction is true, and if '☐' is false the entire disjunction is false.)

Chapter Three

P. 154

5. ——P ∧ Q ├ T ∨ {U ∨ [(Q ∨ V) ∧ (W ∨ P)]}

1 (1) ——P ∧ Q	A
1 (2) ——P	1 ∧E
1 (3) P	2 ——E
1 (4) W ∨ P	3 ∨I
1 (5) Q	1 ∧E
1 (6) Q ∨ V	5 ∨I
1 (7) (Q ∨ V) ∧ (W ∨ P)	6, 4 ∧I
1 (8) U ∨ [(Q ∨ V) ∧ (W ∨ P)]	7 ∨I
1 (9) T ∨ {U ∨ [(Q ∨ V) ∧ (W ∨ P)]}	8 ∨I

10. ——P ∧ ——(———Q ∧ ————R) ├ (P ∧ —Q) ∧ —R

1 (1) ——P ∧ ——(———Q ∧ ————R)	A
1 (2) ——P	1 ∧E
1 (3) P	2 ——E
1 (4) ——(———Q ∧ ————R)	1 ∧E
1 (5) ———Q ∧ ————R	4 ——E
1 (6) ———Q	5 ∧E
1 (7) —Q	6 ——E
1 (8) ———R	5 ∧E
1 (9) —R	8 ——E
1 (10) P ∧ —Q	3, 7 ∧I
1 (11) (P ∧ —Q) ∧ —R	10, 9 ∧I

P. 165

5. P → Q, R → S ├ (Q → R) → (P → S)

1 (1) P → Q	A
2 (2) R → S	A
3 (3) Q → R	A
4 (4) P	A
1,4 (5) Q	1,4 →E
1,3,4 (6) R	3,5 →E
1,2,3,4,(7) S	2,4 →E
1,2,3 (8) P → S	4,7 →I
1,2 (9) (Q → R) → (P → S)	3,8 →I

10. −− [P → −−(Q ∧ R)] ├ (R → S) → [P → (Q ∧ S)]

1	(1) −− [P → −−(Q ∧ R)]	A
2	(2) R → S	A
3	(3) P	A
1	(4) P → −−(Q ∧ R)	1 −−E
1,3	(5) −−(Q ∧ R)	4,3 →E
1,3	(6) Q ∧ R	5 −−E
1,3	(7) Q	6 ∧E
1,3	(8) R	6 ∧E
1,2,3	(9) S	2,8 →E
1,2,3	(10) Q ∧ S	7,9 ∧I
1,2	(11) P → (Q ∧ S)	3,10 →I
1	(12) (R → S) → [P → (Q ∧ S)]	2,11 →I

15. (P → S) ∧ (S → Q) ├ [(P → Q) → R] →R

1	(1) (P → S) ∧ (S → Q)	A
1	(2) P → S	1 ∧E
1	(3) S → Q	1 ∧E
4	(4) (P → Q) → R	A
5	(5) P	A
1,5	(6) S	2,5 →E
1,5	(7) Q	3,6 →E
1	(8) P → Q	5,7 →I
1,4	(9) R	4,8 →E
1	(10) [(P → Q) → R] →R	4,9 →I

Note: Once you assume '(P → Q) → R', with the intention of generating 'R', your strategy should be to generate 'P → Q'. Why so? So that from '(P → Q) → R' and 'P → Q' you'll have 'R' by →E. To generate 'P → Q', obviously you'll assume 'P' and aim for 'Q'.

P. 168

5. $P \rightarrow [(Q \rightarrow R) \wedge (R \rightarrow Q)], [(Q \rightarrow R) \wedge (R \rightarrow Q)] \rightarrow P \vdash P \leftrightarrow (Q \leftrightarrow R)$

	1	(1)	$P \rightarrow [(Q \rightarrow R) \wedge (R \rightarrow Q)]$	A
	2	(2)	$[(Q \rightarrow R) \wedge (R \rightarrow Q)] \rightarrow P$	A
	3	(3)	P	A
\	1,3	(4)	$(Q \rightarrow R) \wedge (R \rightarrow Q)$	1,3 $\rightarrow$E
	1,3	(5)	$Q \leftrightarrow R$	4 $\leftrightarrow$ I
	1	(6)	$P \rightarrow (Q \leftrightarrow R)$	3,5 $\rightarrow$I
	7	(7)	$Q \leftrightarrow R$	A
	7	(8)	$(Q \rightarrow R) \wedge (R \rightarrow Q)$	7 $\leftrightarrow$ E
	2,7	(9)	P	2,8 $\rightarrow$E
	2	(10)	$(Q \leftrightarrow R) \rightarrow P$	7,9 $\rightarrow$I
	1,2	(11)	$[P \rightarrow (Q \leftrightarrow R)] \wedge [(Q \leftrightarrow R) \rightarrow P]$	6,10 $\wedge$I
	1,2	(12)	$P \leftrightarrow (Q \leftrightarrow R)$	11 $\leftrightarrow$I

<u>Pp. 173 – 174</u>

3. (P ∧ −−Q) ∨ (P ∨ −−R) ⊢ P ∨ R
Difficult. This is a ∨E derivation within the scope of another ∨E derivation. (Notice that the right disjunct of the premise is itself a disjunction.) Two readily provable variations on this sequent follow.

1 (1) (P ∧ −−Q) ∨ (P ∨ −−R)	A
2 (2) P ∧ −−Q	A
2 (3) P	2 ∧E
2 (4) P ∨ R	3 ∨I
(5) (P ∧ −−Q) → (P ∨ R)	2,4 →I

In line-(5), you've demonstrated that if the *left* disjunct of line-(1) is the case then the conclusion is the case: '(P ∧ −−Q) → (P ∨ R)'. You now need to demonstrate that if the *right* disjunct of line-(1) is the case then the conclusion is the case: '(P ∨ −−R) → (P ∨ R)'. In line-(6), therefore, you assume 'P ∨ −−R' and aim for 'P ∨ R'.

It's because (6) is itself a disjunction that you have to initiate a *second* ∨E derivation at this point. Just treat (6) as the sole premise of a new sequent whose conclusion is (unsurprisingly) 'P ∨ R'. In other words, just pretend that you've been asked to prove the following sequent: 'P ∨ −−R ⊢ P ∨ R'. You will succeed in line-(14), where 'P ∨ R' rests only on (6), i.e., on 'P ∨ −−R'.

6 (6) P ∨ −−R	A
7 (7) P	A
7 (8) P ∨ R	7 ∨I
(9) P → (P ∨ R)	7,8 →I
10 (10) −−R	A
10 (11) R	10 −−E
10 (12) P ∨ R	11 ∨I
(13) −−R → (P ∨ R)	10,12, →I
6 (14) P ∨ R	6,9,13 ∨E

Recall that your goal was to prove that if the *right* disjunct of line-(1) is the case then the conclusion is the case: '(P ∨ −−R) → (P ∨ R)'. So now, of course, you perform →I at line-(15):

(15) (P ∨ −−R) → (P ∨ R)	6,14 →E

In line-(5) you demonstrated that if the *left* disjunct of line-(1) is the case then the conclusion is the case: '(P ∧ −−Q) → (P ∨ R)'. In line-(15) you demonstrated that if the *right* disjunct of line-(1) is the case then the conclusion is the case: '(P ∨ −−R) → (P ∨ R)'. Game over.

1 (16) P ∨ R	1,5,15 ∨E

Notice that turning a wedge into a caret turns a difficult derivation into a much easier (but much less interesting) one—which (of course) you should now prove:

3 (variation). (P ∧ — —Q) ∨ (P ∧ — —R) ⊢ P ∨ R

Notice too that throwing a dash into the right disjunct turns one easy derivation into another easy (but ever so slightly more interesting) derivation—which (of course) you should now prove:

3 (variation on the variation). (P ∧ — —Q) ∨ (—P ∧ — —R) ⊢ P ∨ R

5. P → (Q ∨ R), (S ∨ R) → (U ∧ T), (S ∨ Q) → (T ∧ U) ⊢ P → (T ∨ U)

1	(1)	P → (Q ∨ R)	A
2	(2)	(S ∨ R) → (U ∧ T)	A
3	(3)	(S ∨ Q) → (T ∧ U)	A
4	(4)	P	A
1,4	(5)	Q ∨ R	1,4 →E
6	(6)	Q	A
6	(7)	S ∨ Q	6 ∨I
3,6	(8)	T ∧ U	3,7 →E
3,6	(9)	T	8 ∧E
3,6	(10)	T ∨ U	9 ∨I
3	(11)	Q → (T ∨ U)	6,10 →I
12	(12)	R	A
12	(13)	S ∨ R	12 ∨I
2,12	(14)	U ∧ T	2,12 →E
2,12	(15)	U	14 ∧E
2,12	(16)	T ∨ U	15 ∨I
2	(17)	R → (T ∨ U)	12,16 →I
1,2,3,4	(18)	T ∨ U	5,11,17 ∨E
1,2,3	(19)	P → (T ∨ U)	4,18 →I

6. [(Q ∨ S) ∨ R] → T ⊢ P → {[(Q ∨ R) ∨ S] → T}
Much too difficult. Do *not* attempt to prove this sequent until after you've proved either S73, i.e., exercise number 9 on p. 197, or S96, i.e., exercise number 2 on p. 238… Prove the following variation instead:

6 (variation). [(Q ∨ S) ∨ R] → T ⊢ [(Q ∨ R) ∨ S] → T

8. (P ∨ Q) ∧ (P ∨ R) ⊢ P ∨ (Q ∧ R)

Difficult. Another ∨E proof within the scope of another ∨E proof.

1	(1)	(P ∨ Q) ∧ (P ∨ R)	A
1	(2)	P ∨ Q	1 ∧E
3	(3)	P	A
3	(4)	P ∨ (Q ∧ R)	3 ∨I
	(5)	P → [P ∨ (Q ∧ R)]	3,4 →I
6	(6)	Q	A
1	(7)	P ∨ R	1 ∧E
8	(8)	P	A
8	(9)	P ∨ (Q ∧ R)	8 ∨I
	(10)	P → [P ∨ (Q ∧ R)]	8,9 →I
11	(11)	R	A
6,11	(12)	Q ∧ R	6,11 ∧I
6,11	(13)	P ∨ (Q ∧ R)	12 ∨I
6	(14)	R → [P ∨ (Q ∧ R)]	11,13 →I
1,6	(15)	P ∨ (Q ∧ R)	7,10,14 ∨E
1	(16)	Q → [P ∨ (Q ∧ R)]	6,15 →E
1	(17)	P ∨ (Q ∧ R)	2,5,16 ∨E

10. P ∨ (Q ∨ R) ⊢ (P ∨ Q) ∨ R

1	(1)	P ∨ (Q ∨ R)	A
2	(2)	P	A
2	(3)	P ∨ Q	2 ∨I
2	(4)	(P ∨ Q) ∨ R	3 ∨I
	(5)	P → [(P ∨ Q) ∨ R]	2,4 →I
6	(6)	Q ∨ R	A
7	(7)	Q	A
7	(8)	P ∨ Q	7 ∨I
7	(9)	(P ∨ Q) ∨ R	8 ∨I
	(10)	Q → [(P ∨ Q) ∨ R]	7,9 →I
11	(11)	R	A
11	(12)	(P ∨ Q) ∨ R	11 ∨I
	(13)	R → [(P ∨ Q) ∨ R]	11,12 →I
6	(14)	(P ∨ Q) ∨ R	6,10,13 ∨E
	(15)	(Q ∨ R) → [(P ∨ Q) ∨ R]	6,14 →E
1	(16)	(P ∨ Q) ∨ R	1,5,15 ∨E

13. P v [Q v (R v S)] ⊢ [P v (Q v R)] v S
Ridiculously difficult.

1 (1) P v [Q v (R v S)]	A
2 (2) P	A
2 (3) P v (Q v R)	2 vI
2 (4) [P v (Q v R)] v S	3 vI
(5) P → {[P v (Q v R)] v S}	2,4 →I
6 (6) Q v (R v S)	A
7 (7) Q	A
7 (8) Q v R	7 vI
7 (9) P v (Q v R)	8 vI
7 (10) [P v (Q v R)] v S	9 vI
(11) Q → {[P v (Q v R)] v S}	7,10 →I
12 (12) R v S	A
13 (13) R	A
13 (14) Q v R	13 vI
13 (15) P v (Q v R)	14 vI
13 (16) [P v (Q v R)] v S	15 vI
(17) R → {[P v (Q v R)] v S}	13,16 →I
18 (18) S	A
18 (19) [P v (Q v R)] v S	18 vI
(20) S → {[P v (Q v R)] v S}	18,19 →I
12 (21) [P v (Q v R)] v S	12,17,20 vE
(22) (R v S) → {[P v (Q v R)] v S}	12,21 →I
6 (23) [P v (Q v R)] v S	6,11,22 vE
(24) [Q v (R v S)] → {[P v (Q v R)] v S}	6,23 →I
1 (25) [P v (Q v R)] v S	1,5,24 vE

15. Q → (R ∧ S), —R → (S ∧ T) ⊢ P → [(Q v —R) → S]
Much too difficult. Do *not* attempt to prove this sequent until after you've proved either S73, i.e., exercise number 9 on p. 197, or S96, i.e., exercise number 2 on p. 238… Prove the following variation instead:

15 (variation). Q → (R ∧ S), —R → (S ∧ T) ⊢ (Q v —R) → S

16. [S v (R v P)] → T ⊢ (P → Q) → [(R v S) → T]
Much too difficult. Do *not* attempt to prove this sequent until after you've proved either S73, i.e., exercise number 9 on p. 197, or S96, i.e., exercise number 2 on p. 238… Prove the following variation instead:

16 (variation). [S v (R v P)] → T ⊢ (R v S) → T

<u>Pp. 196 – 197</u>

5. P ∧ Q ⊢ −(−P ∨ −Q)

1 (1) P ∧ Q	A
1 (2) P	1 ∧E
1 (3) Q	1 ∧E
4 (4) −P ∨ −Q	A
5 (5) −P	A
1,5 (6) P ∧ −P	2,5 ∧I
5 (7) −(P ∧ Q)	1,6 −I
(8) −P → −(P ∧ Q)	5,7 →I
9 (9) −Q	A
1,9 (10) Q ∧ −Q	3,9 ∧I
9 (11) −(P ∧ Q)	1,10 −I
(12) −Q → −(P ∧ Q)	9,11 →I
4 (13) −(P ∧ Q)	4,8,12 ∨E
1,4 (14) (P ∧ Q) ∧ −(P ∧ Q)	1,13 ∧I
1 (15) −(−P ∨ −Q)	4,14 −I

9. —P ∨ Q ├ P → Q

Very difficult—but it wouldn't be nearly as difficult if you were to prove the very next sequent, number 10, first. So if you're up for a challenge, prove number 10 now; look *very* closely at the sequent itself, i.e., '—(P ∧ —Q) ├ P → Q'; look *very* closely at your *proof* of the sequent; and then return to this sequent, number 9, and prove it.

1	(1)	—P ∨ Q	A
2	(2)	P ∧ —Q	A
2	(3)	P	2 ∧E
2	(4)	—Q	2 ∧E
5	(5)	—P	A
2,5	(6)	P ∧ —P	3,5 ∧I
5	(7)	—(P ∧ —Q)	2,6 —I
	(8)	—P → —(P ∧ —Q)	5,7 →I
9	(9)	Q	A
2,9	(10)	Q ∧ —Q	9,4 ∧I
9	(11)	—(P ∧ —Q)	2,10 —I
	(12)	Q → —(P ∧ —Q)	9,11 →I
1	(13)	—(P ∧ —Q)	1,8,12 ∨E
		.	
		.	
		P → Q	

The tricky aspect of this sequent is recognizing that you need to discover a *wff* that you can generate from '—P ∨ Q', and from which you can then generate 'P → Q'. The obvious candidate would be a *wff* that is logically equivalent to '—P ∨ Q' and 'P → Q'—and that happens to be of a very different form. Hence the choice of '—(P ∧ —Q)'. I stopped short of completing this particular derivation because you'll be deriving 'P → Q' from '—(P ∧ —Q)' in the very next, i.e., the tenth, sequent in this set of derivations.

15. (P ∧ P) → Q, (P → Q) → R ├ (—P → —R) → Q

1	(1)	(P ∧ P) → Q	A
2	(2)	(P → Q) → R	A
3	(3)	—P → —R	A
4	(4)	—P	A
3,4	(5)	—R	3,4 →E
6	(6)	P	A
6	(7)	P ∧ P	6,6 ∧I
1,6	(8)	Q	1,7 →E
1	(9)	P → Q	6,8 →I
1,2	(10)	R	2,9 →E
1,2,3,4	(11)	R ∧ —R	10,5 ∧I
1,2,3	(12)	— —P	4,11 —I
1,2,3	(13)	P	12 — —E
1,2,3	(14)	P ∧ P	13,13 ∧I
1,2,3	(15)	Q	1,14 →E
1,2	(16)	(—P → —R) → Q	3,15 →I

The strategy here is a bit (!) tricky. Assume '—P → —R' in line (3) in order to generate 'Q' in your second-to-last line. You see 'Q' as the consequent of your line-(1) conditional: your target now is 'P ∧ P' in your third-to-last line, the antecedent of your line-(1) conditional. To generate 'P ∧ P', all you need to generate is 'P'; hence your fourth-to-last line. To generate 'P', assume '—P', its opposite, in line (4), and aim for a contradiction. (3) and (4) are begging you to generate '—R' in (5). You see 'R' as the consequent of your line-(2) conditional. You are within striking distance of your contradiction: your goal now is to generate 'R'. You take a second look at line (2), and realize that if you could generate 'P → Q', you could generate 'R'. To generate 'P → Q', you assume 'P' in line (6) and aim for 'R'.

16. —(—P ∧ —Q) ├ P ∨ Q
Adhering to the instructions in the book, you should find the proof of this sequent a challenge, albeit a surmountable one.

20. —(P ∧ —Q) ├ P
Ridiculously difficult—but it would be a snap (in a manner of speaking: 13 lines) if you were to re-visit one of the sequents that you proved already in this set of exercises. Question: Which *wff* can you generate from the premise that would be *immensely* helpful in generating the conclusion?

22. —P ∨ Q, —(Q ∧ —R) ├ P → R
Ridiculously difficult—but it would be a snap (in a manner of speaking: a mere 32 lines) if you were to re-visit two of the sequents that you proved already in this set of exercises. Question: Which *wff* can you generate from the first premise, and which from the second, that, taken together, would be *immensely* helpful in generating the conclusion?

Chapter Four

P. 204

5. ├ —(—Q → —P) → (P ∧ —Q)

1	(1)	—(—Q → —P)	A
2	(2)	—(P ∧ —Q)	A
3	(3)	—Q	A
4	(4)	P	A
3,4	(5)	P ∧ —Q	4,3 ∧I
2,3,4	(6)	(P ∧ —Q) ∧ —(P ∧ —Q)	5.2 ∧I
2,3	(7)	—P	4,6 —I
2	(8)	—Q → —P	3,7 →I
1,2	(9)	(—Q → —P) ∧ —(—Q → —P)	8,1 ∧I
1	(10)	— —(P ∧ —Q)	2,9 —I
1	(11)	P ∧ —Q	10 — —E
	(12)	—(—Q → —P) → (P ∧ —Q)	1,11 →I

After introducing your *Reductio* assumption in line (2), aim for either the opposite of line (1) or the opposite of line (2). Aiming for the opposite of line (1), '—Q → —P', seems promising because '—Q → —P' is a conditional. Assume '—Q', its antecedent, and aim for '—P', its consequent.

6. $\vdash -[P \wedge -(Q \rightarrow P)]$

Very difficult—but (yet again) it would be a snap (in a manner of speaking: 17 lines) if you were to re-visit one of the sequents that you proved already in the set of exercises on pp. 196-7.

1	(1) $P \wedge -(Q \rightarrow P)$	A
1	(2) P	1 $\wedge$E
1	(3) $-(Q \rightarrow P)$	1 $\wedge$E
4	(4) $-(Q \wedge -P)$	A
5	(5) Q	A
6	(6) $-P$	A
5,6	(7) $Q \wedge -P$	5,6 $\wedge$I
4,5,6	(8) $(Q \wedge -P) \wedge -(Q \wedge -P)$	7,4 $\wedge$I
4,5	(9) $--P$	6,8 $-$I
4,5	(10) P	9 $--$E
4	(11) $Q \rightarrow P$	5,10 $\rightarrow$I
1,4	(12) $(Q \rightarrow P) \wedge -(Q \rightarrow P)$	11,3 $\wedge$I
1	(13) $--(Q \wedge -P)$	4,12 $-$I
1	(14) $Q \wedge -P$	13 $--$E
1	(15) $-P$	14 $\wedge$E
1	(16) $P \wedge -P$	2,15 $\wedge$I
	(17) $-[P \wedge -(Q \rightarrow P)]$	1,16 $-$I

Question: Which *wff* can you generate from (3), '$-(Q \rightarrow P)$', that would be *immensely* helpful?

Answer: '$Q \wedge -P$'. (Cf. exercise number 14 on p. 197.) From '$Q \wedge -P$' you would derive '$-P$', and you would then have a contradiction, '$P \wedge -P$', resting on (1).

Question: How do you generate '$Q \wedge -P$'?

Answer: First, assume its opposite, '$-(Q \wedge -P)$', as a *Reductio* assumption in line (4). Second, generate '$Q \rightarrow P$' from '$-(Q \wedge -P)$'. (Cf. exercise number 10 on p. 197.) You'll then have a contradiction, '$(Q \rightarrow P) \wedge -(Q \rightarrow P)$', resting at least in part on (4). And *presto!* You'll zap (4) and have '$--(Q \wedge -P)$', followed, of course, by '$Q \wedge -P$'.

7. $\vdash \{(P \vee Q) \wedge [(-R \rightarrow -P) \wedge (-R \rightarrow -Q)]\} \rightarrow R$

A long and challenging—but do-able— $\vee$E *proof*.

8. $\vdash P \leftrightarrow P$

Very easy—but for a bit of a challenge, try to prove the sequent in four lines max.

10. $\vdash (P \wedge Q) \rightarrow (P \leftrightarrow Q)$

Very difficult. Think of this proof as involving four sections: the line-(1) →I assumption that launches the proof; the proof of '$P \rightarrow Q$' (lines (2) through (13)); the proof of '$Q \rightarrow P$' (lines (14) through (25)); and the *denouement* (lines (26) through (28)). I focus on the second section, whose strategy is far from obvious. Your target is '$P \rightarrow Q$'. Assume 'P' in line (2) and aim for 'Q' in the second-to-last line of this section. Introduce '$\neg Q$' as a *Reductio* assumption in line (3), and try to generate a contradiction. Trial and error will rule out the value of aiming for the opposite of either (1) or (2) or (3). But conjoining (2) and (3), i.e., '$P \wedge \neg Q$', and aiming for *its* opposite, i.e., '$\neg(P \wedge \neg Q)$', is another matter. Introduce '$P \wedge \neg Q$' as a *Reductio* assumption in line (5), *even though you already generated it in line (4),* and aim for a contradiction. The rest of the first section of the proof, culminating in '$P \rightarrow Q$', should be self-explanatory. The second section of the proof, culminating in '$Q \rightarrow P$', is a mirror-image of the first section.

1	(1) $P \wedge Q$	A
2	(2) P	A
3	(3) $\neg Q$	A
2,3	(4) $P \wedge \neg Q$	2,3 ∧I
5	(5) $P \wedge \neg Q$	A
5	(6) $\neg Q$	5 ∧E
1	(7) Q	1 ∧E
1,5	(8) $Q \wedge \neg Q$	7,6 ∧I
1	(9) $\neg(P \wedge \neg Q)$	5,8 ¬I
1,2,3	(10) $(P \wedge \neg Q) \wedge \neg(P \wedge \neg Q)$	4,9 ∧I
1,2	(11) $\neg\neg Q$	3,10 ¬I
1,2	(12) Q	11 ¬¬E
1	(13) $P \rightarrow Q$	2,12 →I
14	(14) Q	A
15	(15) $\neg P$	A
14,15	(16) $Q \wedge \neg P$	14,15 ∧I
17	(17) $Q \wedge \neg P$	A
17	(18) $\neg P$	17 ∧E
1	(19) P	1 ∧E
1,17	(20) $P \wedge \neg P$	19,18 ∧I
1	(21) $\neg(Q \wedge \neg P)$	17,20 ¬I
1,14,15	(22) $(Q \wedge \neg P) \wedge \neg(Q \wedge \neg P)$	16,21 ∧I
1,14	(23) $\neg\neg P$	15,22 ¬I
1,14	(24) P	23 ¬¬E
1	(25) $Q \rightarrow P$	14,24 →I
1	(26) $(P \rightarrow Q) \wedge (Q \rightarrow P)$	13,25 ∧I
1	(27) $P \leftrightarrow Q$	26 ↔I

(28) $(P \wedge Q) \rightarrow (P \leftrightarrow Q)$ 1,27→I

Note that the very difficult proof of exercise number 10 (above) will be a snap once you've proved S96, i.e., exercise number 2 on p. 238…

P. 217

5. P

(1) *Every wff* in the column is a substitution-instance of 'P'. ('P', in turn, isn't a substitution-instance of *any* of the *wffs* in the column.)
(2) The following is the S.I.-generator of, e.g., the first *wff:*

P
$[(P \vee P) \vee -(P \vee P)] \wedge -(P \vee P)$

10. $(Q \vee -P) \wedge -S$

(1) The first, second, tenth, and twelfth *wffs* are substitution-instances of '$(Q \vee -P) \wedge -S$'.
(2) The following are the respective S.I.-generators. (For ease of reading, I list the atomics in the order in which they appear in the original *wff,* and *not* in alphabetical order.)

1.

Q	P	S
$P \vee P$	$P \vee P$	$P \vee P$

2.

Q	P	S
$P \wedge Q$	$P \wedge Q$	$-(Q \wedge -R)$

10.

Q	P	S
Q	P	S

I.e., every *wff* is its own substitution-instance.

12.

Q	P	S
$P \vee Q$	$P \vee Q$	$P \vee Q$

<u>Pp. 238-239</u>

3. $(P \wedge Q) \vee (-P \wedge -Q) \vdash P \leftrightarrow Q$
A long and challenging—but do-able—derivation involving two applications of $\vee$E *and several applications of derived rules.*

4. $P \leftrightarrow Q \vdash (P \wedge Q) \vee (-P \wedge -Q)$
Moderately difficult. Two versions follow.

First Version:

1	(1)	$P \leftrightarrow Q$	A
1	(2)	$(P \rightarrow Q) \wedge (Q \rightarrow P)$	1 $\leftrightarrow$E
1	(3)	$P \rightarrow Q$	2 $\wedge$E
1	(4)	$Q \rightarrow P$	2 $\wedge$E
	(5)	$P \vee -P$	EM
6	(6)	P	A
1,6	(7)	Q	3,6 $\rightarrow$E
1,6	(8)	$P \wedge Q$	6,7 $\wedge$I
1,6	(9)	$(P \wedge Q) \vee (-P \wedge -Q)$	8 $\vee$I
1	(10)	$P \rightarrow [(P \wedge Q) \vee (-P \wedge -Q)]$	6,9 $\rightarrow$I
11	(11)	$-P$	A
1,11	(12)	$-Q$	4,11 MT
1,11	(13)	$-P \wedge -Q$	11,12 $\wedge$I
1,11	(14)	$(P \wedge Q) \vee (-P \wedge -Q)$	13 $\vee$I
1	(15)	$-P \rightarrow [(P \wedge Q) \vee (-P \wedge -Q)]$	11,14 $\rightarrow$I
1	(16)	$(P \wedge Q) \vee (-P \wedge -Q)$	5,10,15 $\vee$E

Second Version (Needlessly Long):

1	(1) P ↔ Q	A
1	(2) (P → Q) ∧ (Q → P)	1 ↔E
1	(3) P → Q	2 ∧E
1	(4) Q → P	2 ∧E
5	(5) —[(P ∧ Q) ∨ (—P ∧ —Q)]	A
5	(6) —(P ∧ Q) ∧ — (—P ∧ —Q)	5 DM
5	(7) —(P ∧ Q)	6 ∧E
5	(8) — (—P ∧ —Q)	6 ∧E
5	(9) —P ∨ —Q	7 DM
5	(10) P → —Q	9 MI
5	(11) —P → Q	8 MI
	(12) P ∨ —P	EM
13	(13) P	A
1,13	(14) Q	3,13 →E
15	(15) —Q	A
1,13,15	(16) Q ∧ —Q	14,15 ∧I
1,13	(17) — —Q	15,16 —I
1,5,13	(18) — P	10,17 MT
1,5,13	(19) P ∧ —P	13,18 ∧I
1,5	(20) P → (P ∧ —P)	13,19 →I
21	(21) —P	A
1,21	(22) —Q	4,21 MT
1,5,21	(23) — —P	11,22 MT
1,5,21	(24) P	23 — —E
1,5, 21	(25) P ∧ —P	24,21 ∧I
1,5	(26) —P → (P ∧ —P)	21,25 →I
1,5	(27) P ∧ —P	12,20,26
1	(28) — —[(P ∧ Q) ∨ (—P ∧ —Q)]	5,27 —I
1	(29) (P ∧ Q) ∨ (—P ∧ —Q)	28 — —E

5. −(−P ∨ −Q) ├ P ∧ Q

1	(1)	−(−P ∨ −Q)	A
2	(2)	−P	A
2	(3)	−P ∨ −Q	2 ∨I
1,2	(4)	(−P ∨ −Q) ∧ −(−P ∨ −Q)	3,1 ∧I
1	(5)	− −P	2,4 −I
1	(6)	P	5 − −E
7	(7)	−Q	A
7	(8)	−P ∨ −Q	7 ∨I
1,7	(9)	(−P ∨ −Q) ∧ −(−P ∨ −Q)	8,1 ∧I
1	(10)	− −Q	7,9 −I
1	(11)	Q	10 − −E
1	(12)	P ∧ Q	6,11 ∧I

10. −(−P ∧ −Q) ├ −P → Q

1	(1)	−(−P ∧ −Q)	A
2	(2)	−P	A
3	(3)	−Q	A
2,3	(4)	−P ∧ −Q	2,3 ∧I
1,2,3	(5)	(−P ∧ −Q) ∧ −(−P ∧ −Q)	4,1 ∧I
1,2	(6)	− −Q	3,5 −I
1,2	(7)	Q	6 − −E
1	(8)	−P → Q	2,7 →I

<u>Pp. 241-242</u>

I.5. −P ↔ Q ├ P ↔ −Q

1	(1)	−P ↔ Q	A
1	(2)	(−P → Q) ∧ (Q → −P)	1 ↔E
1	(3)	−P → Q	2 ∧E
1	(4)	Q → −P	2 ∧E
5	(5)	P	A
5	(6)	− −P	5 SI 61
1,5	(7)	−Q	4,6 MT
1	(8)	P → −Q	5,7 →I
9	(9)	−Q	A
1,9	(10)	− −P	3,9 MT
1,9	(11)	P	10 − −E
1	(12)	−Q → P	9,11 →I
1	(13)	(P → −Q) ∧ (−Q → P)	8,12 ∧I
1	(14)	P ↔ −Q	13 ↔I

I.10. (P ↔ Q) ↔ R ├ P ↔ (Q ↔ R)
In what follows, I do *only* the first half of this proof. I leave it to the reader to do the second half.

1 (1) (P ↔ Q) ↔ R	A
2 (2) P	A
3 (3) Q	A
1 (4) [(P ↔ Q) → R] ∧ [R → (P ↔ Q)]	1 ↔E
1 (5) (P ↔ Q) → R	4 ∧E
3 (6) P → Q	3 SI(S) 96 CC
2 (7) Q → P	2 SI 96 CC
2,3 (8) (P → Q) ∧ (Q → P)	6,7 ∧I
2,3 (9) P ↔ Q	8 ↔I
1,2,3 (10) R	5,9 →E
1,2 (11) Q → R	3,10 →I
12 (12) R	A
1 (13) R → (P ↔ Q)	4 ∧E
1,12 (14) P ↔ Q	13,12 →E
1,12 (15) (P → Q) ∧ (Q → P)	14 ↔E
1,12 (16) P → Q	15 ∧E
1,2,12 (17) Q	16,2 →E
1,2 (18) R → Q	12,17 →I
1,2 (19) (Q → R) ∧ (R → Q)	11,18 ∧I
1,2 (20) Q ↔ R	19 ↔I
1 (21) P → (Q ↔ R)	2,20 →I

.
.
(Q ↔ R) → P
[P → (Q ↔ R)] ∧ [(Q ↔ R) → P]
P ↔ (Q ↔ R)

II.5. ├ (—P ∨ Q) ∨ —(P → Q)
Two versions.

First Version:

1 (1) —[(—P ∨ Q) ∨ —(P → Q)]	A
1 (2) —(—P ∨ Q) ∧ — —(P → Q)	1 DM
1 (3) —(—P ∨ Q)	2 ∧E
1 (4) — —(P → Q)	2 ∧E
1 (5) P → Q	4 — —E
1 (6) —P ∨ Q	5 MI
1 (7) (—P ∨ Q) ∧ —(—P ∨ Q)	6,3 ∧I
(8) — —[(—P ∨ Q) ∨ —(P → Q)]	1,7 —I
(9) (—P ∨ Q) ∨ —(P → Q)	8 — —E

Second Version:

	(1) (P → Q) ∨ ─(P → Q)	EM
2	(2) P → Q	A
2	(3) ─P ∨ Q	2 MI
2	(4) (─P ∨ Q) ∨ ─(P → Q)	3 ∨I
	(5) (P → Q) → [(─P ∨ Q) ∨ ─(P → Q)]	2,4 →I
6	(6) ─(P → Q)	A
6	(7) (─P ∨ Q) ∨ ─(P → Q)	6 ∨I
	(8) ─(P → Q) → [(─P ∨ Q) ∨ ─(P → Q)]	6,7 →I
	(9) (─P ∨ Q) ∨ ─(P → Q)	1,5,8 ∨E

II.10. ├ [(─P → Q) ∨ R] → [P ∨ (─Q → R)]

1	(1) ─{[(─P → Q) ∨ R] → [P ∨ (─Q → R)]}	A
1	(2) [(─P → Q) ∨ R] ∧ ─[P ∨ (─Q → R)]	1 NI
1	(3) (─P → Q) ∨ R	2 ∧E
1	(4) ─[P ∨ (─Q → R)]	2 ∧E
1	(5) ─P ∧ ─(─Q → R)]	4 DM
1	(6) ─P	5 ∧E
1	(7) ─(─Q → R)]	5 ∧E
1	(8) ─Q ∧ ─R	7 NI
1	(9) ─Q	8 ∧E
1	(10) ─R	8 ∧E
1	(11) ─P → Q	3,10 DS
1	(12) Q	11,6 →E
1	(13) Q ∧ ─Q	12,9 ∧I
	(14) ─ ─{[(─P → Q) ∨ R] → [P ∨ (─Q → R)]}	1,13 ─I
	(15) {[(─P → Q) ∨ R] → [P ∨ (─Q → R)]}	14 ─ ─E

II.12. $\vdash [\neg(P \to \neg Q) \vee \neg(P \to \neg R)] \to [P \wedge \neg(\neg Q \wedge \neg R)]$

Difficult.

1	(1)	$\neg(P \to \neg Q) \vee \neg(P \to \neg R)$	A
2	(2)	$\neg(P \to \neg Q)$	A
2	(3)	$P \wedge \neg\neg Q$	2 NI
2	(4)	P	3 $\wedge$E
2	(5)	$\neg\neg Q$	3 $\wedge$E
2	(6)	Q	5 $\neg\neg$E
2	(7)	$Q \vee R$	6 $\vee$I
2	(8)	$\neg(\neg Q \wedge \neg R)$	7 DM
2	(9)	$P \wedge \neg(\neg Q \wedge \neg R)$	4,8 $\wedge$I
	(10)	$\neg(P \to \neg Q) \to [P \wedge \neg(\neg Q \wedge \neg R)]$	2,9 $\to$I
11	(11)	$\neg(P \to \neg R)$	A
11	(12)	$P \wedge \neg\neg R$	11 NI
11	(13)	P	12 $\wedge$E
11	(14)	$\neg\neg R$	12 $\wedge$E
11	(15)	R	14 $\neg\neg$E
11	(16)	$Q \vee R$	15 $\vee$I
11	(17)	$\neg(\neg Q \wedge \neg R)$	16 DM
11	(18)	$P \wedge \neg(\neg Q \wedge \neg R)$	13,17 $\wedge$I
	(19)	$\neg(P \to \neg R) \to [P \wedge \neg(\neg Q \wedge \neg R)]$	11,18 $\to$I
1	(20)	$P \wedge \neg(\neg Q \wedge \neg R)$	1,10,19 $\vee$E
	(21)	$[\neg(P \to \neg Q) \vee \neg(P \to \neg R)] \to [P \wedge \neg(\neg Q \wedge \neg R)]$	1,20 $\to$I

II.15. $\vdash \neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]$
Three versions.

First Version:

1	(1)	$P \vee Q$	A
1	(2)	$(R \rightarrow S) \rightarrow (P \vee Q)$	1 CC
	(3)	$(P \vee Q) \rightarrow [(R \rightarrow S) \rightarrow (P \vee Q)]$	1,2 $\rightarrow$I
	(4)	$\neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]$	3 MI

Second Version:

1	(1)	$\neg \{\neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]\}$	A
1	(2)	$\neg\neg(P \vee Q) \wedge \neg[(R \rightarrow S) \rightarrow (P \vee Q)]$	1 DM
1	(3)	$\neg\neg(P \vee Q)$	2 $\wedge$E
1	(4)	$\neg[(R \rightarrow S) \rightarrow (P \vee Q)]$	2 $\wedge$E
1	(5)	$(R \rightarrow S) \wedge \neg(P \vee Q)$	4 NI
1	(6)	$\neg(P \vee Q)$	5 $\wedge$E
1	(7)	$\neg(P \vee Q) \wedge \neg\neg(P \vee Q)$	6,3 $\wedge$I
	(8)	$\neg\neg \{\neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]\}$	1,7 $\neg$I
	(9)	$\neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]$	8 $\neg\neg$E

Third Version:

	(1)	$(P \vee Q) \vee \neg(P \vee Q)$	EM
2	(2)	$P \vee Q$	A
2	(3)	$(R \rightarrow S) \rightarrow (P \vee Q)$	2 CC
2	(4)	$\neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]$	3 $\vee$I
	(5)	$(P \vee Q) \rightarrow \{\neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]\}$	2,4 $\rightarrow$I
6	(6)	$\neg(P \vee Q)$	A
6	(7)	$\neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]$	6 $\vee$I
	(8)	$\neg(P \vee Q) \rightarrow \{\neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]\}$	6,7 $\rightarrow$I
	(9)	$\neg(P \vee Q) \vee [(R \rightarrow S) \rightarrow (P \vee Q)]$	1,5,8 $\vee$E

Chapter Five

Pp. 273-274

5. Either Oedipus or Pogdor isn't vicious unless Monty is vicious.
$-Vm \rightarrow (-Vo \vee -Vp)$
10. No unicorns are silly.
$(\forall x)(Ux \rightarrow -Sx)$ —or— $-(\exists x)(Ux \wedge Sx)$

15. Every unicorn that's either silly or treacherous is vicious.
$(\forall x)\{[Ux \wedge (Sx \vee Tx)] \rightarrow Vx\}$—or— $(\forall x)\{Ux \rightarrow [(Sx \vee Tx) \rightarrow Vx]\}$

20. Not a single rabid, silly, and treacherous unicorn is vicious.
$-(\exists x)(Ux \wedge Rx \wedge Sx \wedge Tx \wedge Vx)$ –or— $(\forall x)[(Ux \wedge Rx \wedge Sx \wedge Tx) \rightarrow -Vx]$

25. Silly unicorns are treacherous.
$(\forall x)[(Ux \wedge Sx) \rightarrow Tx]$

30. Some dragons that are rabid are neither silly nor non-silly.
$(\exists x)[Dx \wedge Rx \wedge -(Sx \vee -Sx)]$

35. Neither dragons nor unicorns are either vicious or treacherous.
$(\forall x)[(Dx \vee Ux) \rightarrow -(Vx \vee Tx)]$
or: $-(\exists x)[(Dx \vee Ux) \wedge (Vx \vee Tx)]$
or: $(\forall x)[Dx \rightarrow -(Vx \vee Tx)] \wedge (\forall x)[Ux \rightarrow -(Vx \vee Tx)$
or: $-(\exists x)[Dx \wedge (Vx \vee Tx)] \wedge -(\exists x)[Ux \wedge (Vx \vee Tx)]$
or: $-\{(\exists x)[Dx \wedge (Vx \vee Tx)] \vee (\exists x)[Ux \wedge (Vx \vee Tx)]\}$

40. There exists no object which is both vicious and non-vicious.
$-(\exists x)(Vx \wedge -Vx)$ —or— $(\forall x)-(Vx \wedge -Vx)$

P. 288

5. False. Interpret 'Cx' to mean 'x is a clown' and 'Tx' to mean 'x is a tyrant'. On the one hand, '$(\exists x)(Tx \wedge Cx)$' means 'Some tyrants are clowns', or 'There exists at least one object x such that x is a tyrant and x is a clown'. On the other hand, '$(\exists x)(Tx \wedge Cy)$' means 'There exists at least one object x such that x is a tyrant—and (by the way) y is a clown'. So interpreted, the two *wffs* have radically different meanings.

10. False. To be bound by a quantifier, a variable *has* to fall within the scope of that quantifier. See the first criterion of the definition of 'bound' on p. 283, and consider the *wff* '$(\forall x)Tx \rightarrow Cx$'.

Pp. 293-294

5. Not all tyrants are bloodthirsty clowns.
$\neg(\forall x)[Tx \rightarrow (Bx \wedge Cx)]$
or: $(\exists x)[Tx \wedge \neg(Bx \wedge Cx)]$

10. Any bloodthirsty tyrant who isn't a clown is doomed.
$(\forall x)[(Tx \wedge Bx \wedge \neg Cx) \rightarrow Dx]$
or: $\neg(\exists x)(Tx \wedge Bx \wedge \neg Cx \wedge \neg Dx)$

15. No pompous clown who is sanctimonious is a doomed non-tyrant.
$(\forall x)\ [(Cx \wedge Px \wedge Sx) \rightarrow \neg(\neg Tx \wedge Dx)]$
or: $\neg(\exists x)\ (Cx \wedge Px \wedge Sx \wedge \neg Tx \wedge Dx)$

20. Neither Ardbeg nor Bobo is a doomed clown.
$\neg[(Ca \wedge Da) \vee (Cb \wedge Db)]$

25. Only clowns aren't doomed.
$(\forall x)(\neg Dx \rightarrow Cx)$ —or— $(\forall x)(\neg Cx \rightarrow Dx)$

30. Neither bloodthirsty clowns nor jolly tyrants exist.
$\neg(\exists x)[(Cx \wedge Bx) \vee (Tx \wedge Jx)]$
or: $\neg(\exists x)(Cx \wedge Bx) \wedge \neg(\exists x)(Tx \wedge Jx)]$
or: $(\forall x)\neg(Cx \wedge Bx) \wedge (\forall x)\neg(Tx \wedge Jx)$

34. If every tyrant is bloodthirsty then Ardbeg is doomed.
$(\forall x)(Tx \rightarrow Bx) \rightarrow Da$
or: $(\exists x)[(Tx \rightarrow Bx) \rightarrow Da]$ (Odd, yes? It won't seem so odd once you've made it through (8a) and (8b) on p. 357.)

Chapter Six

Pp. 304-305

5. Ardbeg doesn't belittle everyone.
$-(\forall x)(Px \rightarrow Bax)$
or: $(\exists x)(Px \wedge -Bax)$

10. Anyone who clobbers either Ardbeg or Bobo is rude.
$(\forall x)\{[(Px \wedge (Cxa \vee Cxb)] \rightarrow Rx\}$

15. No one belittles everyone.
Loglish: $(\forall x)(Px \rightarrow$ it's not the case that x belittles everyone)
$(\forall x)[Px \rightarrow -(\forall y)(Py \rightarrow Bxy)]$

or: $(\forall x)[Px \rightarrow (\exists y)(Py \wedge -Bxy)]$
or: $-(\exists x)[Px \wedge (\forall y)(Py \rightarrow Bxy)]$

20. Someone belittles someone.
Loglish: $(\exists x)(Px \wedge x$ belittle someone)
$(\exists x)[Px \wedge (\exists y)(Py \wedge Bxy)]$

or: $(\exists x)[Px \wedge (\exists y)(Py \wedge Byx)]$
or: $(\exists x)(\exists y)[Px \wedge (Py \wedge Bxy)]$
or: $(\exists x)(\exists y)[Px \wedge (Py \wedge Byx)]$

25. No one clobbers anyone who belittles Ardbeg.
Loglish: $-(\exists x)(Px \wedge x$ clobbers someone who belittles Ardbeg)
$-(\exists x)[Px \wedge (\exists y)(Py \wedge Bya \wedge Cxy)]$

or: $(\forall x)\{Px \rightarrow (\forall y)[(Py \wedge Bya) \rightarrow -Cxy]\}$
or: $(\forall x)\{Px \rightarrow -(\exists y)[(Py \wedge Bya) \wedge Cxy]\}$

30. There exists at least one person who clobbers everyone who belittles anyone.
Loglish$_1$: $(\exists x)(Px \wedge x$ clobbers everyone who belittles anyone)
Loglish$_2$: $(\exists x)[Px \wedge (\forall y)(Py \rightarrow$ if y belittles anyone then $Cxy)]$
$(\exists x)(Px \wedge (\forall y)\{Py \rightarrow (\forall z)[(Pz \wedge Byz) \rightarrow Cxy]\})$

31. No one belittles anyone who clobbers anyone.
Loglish$_1$: $-(\exists x)(Px \wedge x$ belittles someone who clobbers someone)
Loglish$_2$: $-(\exists x)[Px \wedge (\exists y)(Py \wedge y$ clobbers someone $\wedge Bxy)]$
$-(\exists x)(Px \wedge (\exists y)\{Py \wedge (\exists z)[(Pz \wedge Cyz) \wedge Bxy]\})$

35. Some rude people clobber only people.
Loglish: $(\exists x)[(Px \wedge Rx) \wedge x$ clobbers only people]
$(\exists x)[(Px \wedge Rx) \wedge (\forall y)(Cxy \rightarrow Py)]$
or: $(\exists x)[(Px \wedge Rx) \wedge (\forall y)(-Py \rightarrow -Cxy)]$

Pp. 313-314

5. Every unemployed ballerina except Nicole is famous.
$(\forall x)[(Bx \wedge Ux \wedge x \neq n) \rightarrow Fx]$

7. Every ballerina except Nicole reveres every actor except Oedipus.
Loglish: $(\forall x)[(Bx \wedge x \neq n) \rightarrow x$ reveres every actor except Oedipus]
$(\forall x)\{[(Bx \wedge x \neq n) \rightarrow (\forall y)[(Ay \wedge y \neq o) \rightarrow Rxy]\}$

10. No famous actor other than either Monty or Oedipus reveres every cabaret singer except Nicole. *Difficult.*
$-(\exists x)\{Ax \wedge Fx \wedge -(x = m \vee x = o) \wedge (\forall y)[(Cy \wedge y \neq n) \rightarrow Rxy]\}$
or: $(\forall x)\{[Ax \wedge Fx \wedge -(x = m \vee x = o)] \rightarrow -(\forall y)[(Cy \wedge y \neq n) \rightarrow Rxy]\}$
or: $(\forall x)\{[Ax \wedge Fx \wedge -(x = m \vee x = o)] \rightarrow (\exists y)[(Cy \wedge y \neq n) \wedge -Rxy]\}$

P. 317

5. There exist at least two orange kangaroos.
$(\exists x)[Kx \wedge Ox \wedge (\exists y)(Ky \wedge Oy \wedge y \neq x)]$

10. There exist exactly three kangaroos.
$(\exists x)[Kx \wedge (\exists y)(Ky \wedge y \neq x \wedge (\exists z)\{Kz \wedge z \neq x \wedge z \neq y \wedge (\forall w)[Kw \rightarrow (w = x \vee w = y \vee w = z)]\})]$

Pp. 322-324

Section I

5. Ardbeg is a Platonist, and every Platonist other than Ardbeg is a hero.
$Pa \wedge (\forall x)[(Px \wedge x \neq a) \rightarrow Hx]$

10. The Platonist hero didn't slay the dragon.
$(\exists x)(Px \wedge Hx \wedge (\forall y)\{[(Py \wedge Hy) \rightarrow y = x] \wedge (\exists z)[Dz \wedge (\forall w)(Dw \rightarrow w = z) \wedge -Sxz]\})$

12. The hero, who slew the dragon, is a Platonist. *Difficult.*
Revised English$_1$: There's exactly one hero, x, and x slew the dragon, and x is a Platonist.
Revised English$_2$: There's exactly one hero, x, and there's exactly one dragon, z, and x slew z, and x is a Platonist.
Loglish: $(\exists x)\{[Hx \wedge (\forall y)\ (Hy \rightarrow y = x)] \wedge x$ slew the dragon $\wedge\ Px\}$
$(\exists x)([Hx \wedge (\forall y)(Hy \rightarrow y = x)] \wedge (\exists z)\{[Dz \wedge (\forall w)(Dw \rightarrow w = z)] \wedge Sxz\} \wedge Px)$

Note: Strictly speaking, the two sets of square brackets are unnecessary; however, they make for a reading that most closely approximates the grammar of the original English sentence.

15. Every Platonist except Bobo slew exactly two Platonist dragons.
Very difficult. Note the radical difference between the following two attempts to paraphrase (15):

> (a) There exist exactly two Platonist dragons, and every Platonist except Bobo slew them.
> (b) Every Platonist except Bobo slew exactly two Platonist dragons—*regardless of how many Platonist dragons exist.*

Note too that (15) allows for the existence of more than two Platonist dragons. So does (b)—and so *doesn't* (a). It's (b), therefore, and not (a), that captures the sense of (15).

Loglish$_1$: $(\forall x)[(Px \wedge x \neq b) \rightarrow$ (x slew exactly two Platonist dragons; i.e., there exist exactly two Platonist dragons that x slew)]

Loglish$_3$ (for 'there exist exactly two Platonist dragons that x slew'):
$(\exists y)((Dy \wedge Py \wedge Sxy) \wedge (\exists z)\{(Dz \wedge Pz \wedge Sxz) \wedge z \neq y \wedge$
$(\forall w)[(Dw \wedge Pw \wedge Sxw) \rightarrow (w = y \vee w = z)]\})$

$(\forall x)[(Px \wedge x \neq b) \rightarrow (\exists y)((Dy \wedge Py \wedge Sxy) \wedge$
$(\exists z)\{(Dz \wedge Pz \wedge Sxz) \wedge\ z \neq y \wedge (\forall w)[(Dw \wedge Pw \wedge Sxw) \rightarrow (w = y \vee w = z)]\})]$

The devil, as they say, is in the details. What they *don't* say is that the details can be purely punctuational.

<u>Section II</u>

5. Not every natural number is greater than some natural number.
$-(\forall x)[Nx \rightarrow (\exists y)(Ny \wedge Gxy)]$ —or— $(\exists x)[Nx \wedge -(\exists y)(Ny \wedge Gxy)]$

10. Every odd natural number other than 1 is greater than 2.
$(\forall x)[(Nx \wedge x \neq b) \rightarrow Gxc]$

15. *Difficult.* No odd natural numbers are divisible by any even natural numbers, but some even natural numbers are divisible by some odd natural numbers.
$-(\exists x)[(Nx \wedge Ox) \wedge (\exists y)[(Ny \wedge Ey \wedge Dxy)] \wedge (\exists z)[(Nz \wedge Ez \wedge (\exists w)(Nw \wedge Ow \wedge Dzw)]$
or: $(\forall x)[(Nx \wedge Ox) \rightarrow -(\exists y)[(Ny \wedge Ey \wedge Dxy)] \wedge (\exists z)[(Nz \wedge Ez \wedge (\exists w)(Nw \wedge Ow \wedge Dzw)]$

20. No odd natural number is divisible by any natural number that is greater than 1 but less than 3.
$-(\exists x)[(Nx \wedge Ox) \wedge (\exists y)(Ny \wedge Gyb \wedge Lyd \wedge Dxy)]$
or: $(\forall x)[(Nx \wedge Ox) \rightarrow -(\exists y)(Ny \wedge Gyb \wedge Lyd \wedge Dxy)]$
or: $(\forall x)\{(Nx \wedge Ox) \rightarrow (\forall y)[(Ny \wedge Gyb \wedge Lyd) \rightarrow -Dxy)]\}$

25. No prime number other than 2 is divisible by an even number.
$-(\exists x)[Px \wedge x \neq c \wedge (\exists y)(Ny \wedge Ey \wedge Dxy)]$
or: $(\forall x)\{(Px \wedge x \neq c) \rightarrow (\forall y)[(Ny \wedge Ey) \rightarrow -Dxy]\}$
or: $(\forall x)\{(Px \wedge x \neq c) \rightarrow -(\exists y)[(Ny \wedge Ey) \wedge Dxy]\}$

<u>Pp. 344-345</u>

<u>Section I</u>

<u>Domain: People</u>

2. $\underline{-(\forall x)(Lx \rightarrow -Mx)}$
$\therefore -(\exists x)(Lx \wedge -Mx)$

<u>It's not the case that no logicians are magicians; i.e., some logicians are magicians.</u>
There doesn't exist a single logician who isn't a magician; i.e., all logicians are magicians.

5. $\underline{(\forall x)(\exists y)Ryx}$
$\therefore (\forall x)(\exists y)Rxy$

<u>Everyone is respected by someone or other.</u>
Therefore everyone respects someone or other.

7. $\underline{-(\exists x)(\forall y)-Rxy}$
$\therefore -(\forall x)(\exists y)-Ryx$

<u>It's not the case that there exists someone who respects no one; i.e., no one respects no one; i.e., everyone respects someone.</u>
Therefore not everyone is such that at least one person fails to respect him (or her); i.e., at least one person is respected by everyone.

QN enables us to see *exactly* what each sentence means: the premise is equivalent to '$(\forall x)(\exists y)Rxy$' and the conclusion is equivalent to '$(\exists x)(\forall y)Ryx$'.

10. $\underline{(\exists x)-(\forall y)Rxy}$
$\therefore (\forall x)(\exists y)-Rxy$

<u>There exists at least one person who doesn't respect everyone.</u>
Everyone fails to respect at least one person.

Section II

2. $\underline{-(\forall x)(Lx \rightarrow -Mx)}$
$\therefore -(\exists x)(Lx \wedge -Mx)$

$-[(La \rightarrow -Ma) \wedge (Lb \rightarrow -Mb)]$
T T F FT F T T TF
$-[(La \wedge -Ma) \vee (Lb \wedge -Mb)]$
F T F FT T T T TF

Invalid. One counterexample:

La	*Ma*	*Lb*	*Mb*
T	T	T	F

5. $\underline{(\forall x)(\exists y)Ryx}$
$\therefore (\forall x)(\exists y)Rxy$

$\underline{(\exists y)Rya \wedge (\exists y)Ryb}$
$(\exists y)Ray \wedge (\exists y)Rby$

$(Raa \vee Rba) \wedge (Rab \vee Rbb)$
F T T T F T T
$(Raa \vee Rab) \wedge (Rba \vee Rbb)$
F F F F T T T

$(Raa \vee Rba) \wedge (Rab \vee Rbb)$
T T F T T T F
$(Raa \vee Rab) \wedge (Rba \vee Rbb)$
T T T F F F F

Invalid. Two counterexamples:

Raa	*Rba*	*Rab*	*Rbb*
F	T	F	T
T	F	T	F

The argument, once again, is:

Everyone is respected by someone or other.
Therefore everyone respects someone or other.

First counterexample: a two-person domain:
The premise is true because Ardbeg and Bobo are both respected by Bobo.
The conclusion is false because Ardbeg respects neither himself nor Bobo.

Second counterexample:
Obvious.

7. $\underline{-(\exists x)(\forall y)-Rxy}$
$\therefore -(\forall x)(\exists y)-Ryx$

$\underline{-[(\forall y)-Ray \vee (\forall y)-Rby\ ]}$
$-[(\exists y)-Rya \wedge (\exists y)-Ryb]$

$-[(-Raa \wedge -Rab) \vee (-Rba \wedge -Rbb)]$
T T F   F F T   F F T   F T F
$-[(-Raa \vee -Rba\ ) \wedge (-Rab \vee -Rbb)]$
F T F   T F T   T F T T T F

$-[(-Raa \wedge -Rab) \vee (-Rba \wedge -Rbb)]$
T F T   F T F   F T F   F F T
$-[(-Raa \vee -Rba\ ) \wedge (-Rab \vee -Rbb)]$
F F T   T T F   T T F   T F T

Invalid. Two counterexamples:

Raa	*Rab*	*Rba*	*Rbb*
F	T	T	F
T	F	F	T

The argument, once again, is:

<u>It's not the case that there exists someone who respects no one.</u>
Therefore not everyone is such that at least one person fails to respect him (or her).

First counterexample: a two-person domain:

The premise is true because Ardbeg doesn't respect himself although he respects Bobo, and Bobo respects Ardbeg although she doesn't respect herself.
The conclusion is false; i.e., everyone *is* such that at least one person fails to respect him (or her). Ardbeg fails to respect himself (although he respects Bobo), and Bobo fails to respect herself (although she respects Ardbeg).
Second counterexample:
Obvious.

10. $\underline{(\exists x)\neg(\forall y)Rxy}$
$\therefore (\forall x)(\exists y)\neg Rxy$

$\underline{\neg(\forall y)Ray \lor \neg(\forall y)Rby}$
$(\exists y)\neg Ray \land (\exists y)\neg Rby$

$\neg(Raa \land Rab) \lor \neg(Rba \land Rbb)$
F T T T T T T F F
$(\neg Raa \lor \neg Rab) \land (\neg Rba \lor \neg Rbb)$
F T F F T F F T T T F

$\neg(Raa \land Rab) \lor \neg(Rba \land Rbb)$
F T T T T T F F T
$(\neg Raa \lor \neg Rab) \land (\neg Rba \lor \neg Rbb)$
F T F F T F T F T F T

$\neg(Raa \land Rab) \lor \neg(Rba \land Rbb)$
F T T T T T F F F
$(\neg Raa \lor \neg Rab) \land (\neg Rba \lor \neg Rbb)$
F T F F T F T F T T F

Invalid. Six counterexamples: I present the first three and leave it to the reader to formulate the last three.

Raa	*Rab*	*Rba*	*Rbb*
T	T	T	F
T	T	F	T
T	T	F	F

P. 360

II. 5. $(\forall x)\ \{[Px \rightarrow (\forall y)My] \rightarrow (\exists z)(Mz \rightarrow Nz)\}$

Into Prenex Form:

(1) $(\forall x)\{[Px \rightarrow (\forall y)My] \rightarrow (\exists z)(Mz \rightarrow Nz)\}$
(2) $(\forall x)[(\forall y)(Px \rightarrow My) \rightarrow (\exists z)(Mz \rightarrow Nz)]$ 1 QS (6a to 6b)
(3) $(\forall x)(\exists y)[(Px \rightarrow My) \rightarrow (\exists z)(Mz \rightarrow Nz)]$ 2 QS (8a to 8b)
(4) $(\forall x)(\exists y)\ (\exists z)[(Px \rightarrow My) \rightarrow (Mz \rightarrow Nz)]$ 3 QS (3a to 3b)

Into Purified Form:

(1) $(\forall x)\{[Px \rightarrow (\forall y)My] \rightarrow (\exists z)(Mz \rightarrow Nz)\}$
(2) $(\exists x)[Px \rightarrow (\forall y)My] \rightarrow (\exists z)(Mz \rightarrow Nz)$ 1 QS (7b to 7a)
(3) $[(\forall x)Px \rightarrow (\forall y)My] \rightarrow (\exists z)(Mz \rightarrow Nz)$ 2 QS (8b to 8a)

III. 5. $(\exists z)(\exists x)(\forall y)[Lx \vee (Mz \rightarrow Ny)]$

Into Purified Form:

(1) $(\exists z)(\exists x)(\forall y)[Lx \vee (Mz \rightarrow Ny)]$
(2) $(\exists z)(\exists x)[Lx \vee (\forall y)(Mz \rightarrow Ny)]$ 1 QS (4b to 4a)
(3) $(\exists z)(\exists x)[Lx \vee (Mz \rightarrow (\forall y)Ny)]$ 2 QS (6b to 6a)
(4) $(\exists z)[(\exists x)Lx \vee (Mz \rightarrow (\forall y)Ny)]$ 3 QS (CM, 1b to 1a, CM)
(5) $(\exists x)Lx \vee (\exists z)(Mz \rightarrow (\forall y)Ny)$ 4 QS (1b to 1a)

The move from (3) to (4) requires an instance of Commutation on (3) first, within the square brackets, followed by QS, followed by another instance of Commutation, also within the (expanded) square brackets.

IV. 5. $[(\forall x)Lx \rightarrow (\exists y)My] \rightarrow (\exists z)Nz$

Into Prenex Form:

(1) $[(\forall x)Lx \rightarrow (\exists y)My] \rightarrow (\exists z)Nz$
(2) $(\exists y)[(\forall x)Lx \rightarrow My] \rightarrow (\exists z)Nz$ 1 QS (3a to 3b)
(3) $(\exists y)(\exists x)(Lx \rightarrow My) \rightarrow (\exists z)Nz$ 2 QS (8a to 8b)
(4) $(\exists z)[(\exists y)(\exists x)(Lx \rightarrow My) \rightarrow Nz]$ 3 QS (3a to 3b)
(5) $(\exists z)(\forall y)[(\exists x)(Lx \rightarrow My) \rightarrow Nz]$ 4 QS (7a to 7b)
(6) $(\exists z)(\forall y)(\forall x)[(Lx \rightarrow My) \rightarrow Nz]$ 5 QS (7a to 7b)

Chapter Seven

Pp. 379-380

4. This derivation is *long* (twenty-one lives using only the primitive rules), but you should find it only *slightly* challenging. Plan on making use of $\neg$I (twice), $\vee$E, and $\exists$E.

5. $(\forall x)(Lx \rightarrow Mx) \vdash (\forall x)Lx \rightarrow (\forall x)Mx$

1	(1) $(\forall x)(Lx \rightarrow Mx)$	A
2	(2) $(\forall x)Lx$	A
1	(3) $Lx \rightarrow Mx$	1 $\forall$E
2	(4) Lx	2 $\forall$E
1,2	(5) Mx	3,4 $\rightarrow$E
1,2	(6) $(\forall x)Mx$	5 $\forall$I
1	(7) $(\forall x)Lx \rightarrow (\forall x)Mx$	2,6 $\rightarrow$I

10. $\neg(\exists x)Lx \vdash (\forall x)\neg Lx$

1	(1) $\neg(\exists x)Lx$	A
2	(2) Lx	A
2	(3) $(\exists x) Lx$	2 $\exists$I
1,2	(4) $(\exists x) Lx \wedge \neg(\exists x)Lx$	3,1 $\wedge$I
1	(5) $\neg Lx$	2,4 $\neg$I
1	(6) $(\forall x)\neg Lx$	5 $\forall$I

15. $(\forall x)(Lx \rightarrow \neg Mx) \vdash \neg(\exists x)(Lx \wedge Mx)$

1	(1) $(\forall x)(Lx \rightarrow \neg Mx)$	A
2	(2) $(\exists x)(Lx \wedge Mx)$	A
3	(3) $Lx \wedge Mx$	A
3	(4) Lx	3 $\wedge$E
3	(5) Mx	3 $\wedge$E
1	(6) $Lx \rightarrow \neg Mx$	1 $\forall$E
1,3	(7) $\neg Mx$	6,4 $\rightarrow$E
1,3	(8) $Mx \wedge \neg Mx$	5,7 $\wedge$I
3	(9) $\neg(\forall x)(Lx \rightarrow \neg Mx)$	1,8 $\neg$I
	(10) $(Lx \wedge Mx) \rightarrow \neg(\forall x)(Lx \rightarrow \neg Mx)$	3,9 $\rightarrow$I
2	(11) $\neg(\forall x)(Lx \rightarrow \neg Mx)$	2,10 $\exists$E
1,2	(12) $(\forall x)(Lx \rightarrow \neg Mx) \wedge \neg(\forall x)(Lx \rightarrow \neg Mx)$	1,11 $\wedge$I
1	(13) $\neg(\exists x)(Lx \wedge Mx)$	2,12 $\neg$I

17. A hint: once you've introduced your $\neg$I assumption, do nothing until you've determined which contradiction you should be aiming for.

20. —(∃*x*)(*Lx* ∧—*Mx*) ⊢ (∀*x*)(*Lx* → *Mx*)

1	(1) —(∃*x*)(*Lx* ∧—*Mx*)	A
2	(2) *Lx*	A
3	(3) —*Mx*	A
2,3	(4) *Lx* ∧ — *Mx*	2,3 ∧I
2,3	(5) (∃*x*)(*Lx* ∧—*Mx*)	4 ∃I
1,2,3	(6) (∃*x*)(*Lx* ∧—*Mx*) ∧ —(∃*x*)(*Lx* ∧—*Mx*)	5,1 ∧I
1,2	(7) — —*Mx*	3,6 —I
1,2	(8) *Mx*	7 — —E
1	(9) *Lx* → *Mx*	2,8 →I
1	(10) (∀*x*)(*Lx* → *Mx*)	9 ∀I

<u>P. 414</u>

Please note: The exercises in the first set of exercises on p. 414 appear earlier in the text; my apology…

S182: (∀*x*)*Lx* ⊢ —(∃*x*)—*Lx*

1	(1) (∀*x*)*Lx*	A
2	(2) (∃*x*)—*Lx*	A
3	(3) —*Lx*	A
1	(4) *Lx*	1 ∀E
1,3	(5) *Lx* ∧ —*Lx*	4,3 ∧I
3	(6) —(∀*x*)*Lx*	1,5 —I
	(7) —*Lx* → —(∀*x*)*Lx*	3,6 →I
2	(8) —(∀*x*)*Lx*	2,7 ∃E
1,2	(9) (∀*x*)*Lx* ∧ —(∀*x*)*Lx*	1,8 ∧I
1	(10) —(∃*x*)—*Lx*	2,9 —I

S187: $(\forall x)[Lx \rightarrow (\forall y)My] \vdash (\exists x)Lx \rightarrow (\forall y)My$

1	(1)	$(\forall x)[Lx \rightarrow (\forall y)My]$	A
2	(2)	$(\exists x)Lx$	A
3	(3)	Lx	A
1	(4)	$Lx \rightarrow (\forall y)My$	1 ∀E
1,3	(5)	$(\forall y)My$	4,3 →E
1	(6)	$Lx \rightarrow (\forall y)My$	3,5 →I
1,2	(7)	$(\forall y)My$	2,6 ∃E
1	(8)	$(\exists x)Lx \rightarrow (\forall y)My$	2,7 →I

Question: Why can't you wrap up the ∃E part of your proof at line (5), and off to the right write '2, 4 ∃E'?

Answer: In an ∃E-proof, your goal is to generate the conditional whose antecedent is your ∃E-assumption and whose consequent is your target. Moreover, your ∃E-assumption must play a role in generating this conditional. Note that your line-(3) ∃E assumption plays *no* role in generating (4), whereas it most certainly *does* play a role in generating (6). (Notice the *wffs* cited off to the right of (4) and (6), respectively.)

Chapter Eight

Except when instructed otherwise, be prepared to use any of the various instances of QN *frequently* from this point on. Each of them is such that either *you* have proved it as an exercise or *I* have proved it as an example:

S125: $-(\forall x)Lx \vdash (\exists x)-Lx$
S135: $-(\exists x)Lx \vdash (\forall x)-Lx$
S136: $-(\forall x)-Lx \vdash (\exists x)Lx$
S137: $-(\exists x)-Lx \vdash (\forall x)Lx$
S182: $(\forall x)Lx \vdash -(\exists x)-Lx$
S181: $(\exists x)Lx \vdash -(\forall x)-Lx$
S180: $(\forall x)-Lx \vdash -(\exists x)Lx$
S179: $(\exists x)-Lx \vdash -(\forall x)Lx$

Intuitively, it should strike you that if S125 is a legitimate instance of QN, so is each of the following three sequents:

$-(\forall x)Mx \vdash (\exists x)-Mx$
$-(\forall x)(Lx \wedge Mx) \vdash (\exists x)-(Lx \wedge Mx)$
$-(\forall x)(\exists y)Lxy \vdash (\exists x)-(\exists y)Lxy$

Also intuitively, it should strike you that the *reason* why each of these three sequents is a legitimate instance of QN is that each of these three is a *substitution-instance* of S125. To be sure, I haven't defined, and won't be defining, the notion of a substitution instance for Predicate Logic/Quantification Theory: defining such a notion is a bit tricky. (See Chapter 8, section 4, p. 442: "Odds and Ends".) Even without a definition, however, your intuitions should be up to the task of identifying the relevant substitution-instances in each of the Chapter 8 exercises.

Pp. 427-428

1. S195: $(\exists y)Ly \rightarrow (\exists x)Mx \vdash (\exists x)[(\exists y)Ly \rightarrow Mx]$

1	(1) $(\exists y)Ly \rightarrow (\exists x)Mx$	A
2	(2) $-(\exists x)[(\exists y)Ly \rightarrow Mx]$	A
2	(3) $(\forall x)-[(\exists y)Ly \rightarrow Mx]$	2 QN
2	(4) $-[(\exists y)Ly \rightarrow Mx]$	3 ∀E
2	(5) $(\exists y)Ly \wedge -Mx$	4 NI
2	(6) $(\exists y)Ly$	5 ∧E
2	(7) $-Mx$	5 ∧E
2	(8) $(\forall x)-Mx$	7 ∀I
2	(9) $-(\exists x)Mx$	8 QN
1,2	(10) $-(\exists y)Ly$	1,9 MT
1,2	(11) $(\exists y)Ly \wedge -(\exists y)Ly$	6,10 ∧I
1	(12) $--(\exists x)[(\exists y)Ly \rightarrow Mx]$	2,11 −I
1	(13) $(\exists x)[(\exists y)Ly \rightarrow Mx]$	12 −−E

Note the two uses of QN, the second in a counter-intuitive direction.

1. S196: $(\exists x)[Lx \rightarrow (\forall y)My] \vdash (\forall y)Ly \rightarrow (\forall x)Mx$

1	(1) $(\exists x)[Lx \rightarrow (\forall y)My]$	A
2	(2) $(\forall y)Ly$	A
3	(3) $Lx \rightarrow (\forall y)My$	A
2	(4) Lx	2 ∀E
2,3	(5) $(\forall y)My$	3,4 →E
2,3	(6) My	5 ∀E
2,3	(7) $(\forall x)Mx$	6 ∀I
2	(8) $[Lx \rightarrow (\forall y)My] \rightarrow (\forall x)Mx$	3,7 →I
1,2	(9) $(\forall x)Mx$	1,8 ∃E
1	(10) $(\forall y)Ly \rightarrow (\forall x)Mx$	2,9 →I

Note the necessity of de-quantifying (5) in a variable other than '*x*'. '*My*' in line (6) is a perfectly suitable candidate for ∀I, because '*y*' doesn't appear free in either (2) or (3), the *wffs* on which (6) rests. On the other hand, if (6) had been '*Mx*', it would have been a perfectly *unsuitable* candidate for ∀I, precisely because '*x*' *does* appear free in (3).

2. Please note: S198, S199, S200, and S201 have been proved already in the book; my apology….

3. S206: $(\exists x)(\forall y)-Lxy \vdash -(\forall x)(\exists y)Lxy$

Domain: People. ‘There’s at least one person who doesn’t love anyone. Therefore not everyone loves someone.’

Without QN:

1	(1)	$(\exists x)(\forall y)-Lxy$	A
2	(2)	$(\forall y)-Lxy$	A
2	(3)	$-Lxy$	2 ∀E
4	(4)	$(\forall x)(\exists y)Lxy$	A
4	(5)	$(\exists y)Lxy$	4 ∀E
6	(6)	Lxy	A
2,6	(7)	$Lxy \wedge -Lxy$	6,3 ∧I
6	(8)	$-(\forall y)-Lxy$	2,7 —I
	(9)	$Lxy \rightarrow -(\forall y)-Lxy$	6,8 →I
4	(10)	$-(\forall y)-Lxy$	5,9 ∃E
2,4	(11)	$(\forall y)-Lxy \wedge -(\forall y)-Lxy$	2,10 ∧I
2	(12)	$-(\forall x)(\exists y)Lxy$	4,11 —I
	(13)	$(\forall y)-Lxy \rightarrow -(\forall x)(\exists y)Lxy$	2,12 →I
1	(14)	$-(\forall x)(\exists y)Lxy$	1,13 ∃E

With QN:

1	(1)	$(\exists x)(\forall y)-Lxy$	A
2	(2)	$(\forall y)-Lxy$	A
2	(3)	$-(\exists y)Lxy$	2 QN
4	(4)	$(\forall x)(\exists y)Lxy$	A
4	(5)	$(\exists y)Lxy$	4 ∀E
2,4	(6)	$(\exists y)Lxy \wedge -(\exists y)Lxy$	5,3 ∧I
2	(7)	$-(\forall x)(\exists y)Lxy$	4,6 —I
	(8)	$(\forall y)-Lxy \rightarrow -(\forall x)(\exists y)Lxy$	2,7 →I
1	(9)	$-(\forall x)(\exists y)Lxy$	1,8 ∃E

With QN, if you were permitted to operate on *wff*-fragments:

1	(1)	$(\exists x)(\forall y)-Lxy$	A
1	(2)	$(\exists x)-(\exists y)Lxy$	1 QN (on a fragment of (1))
1	(3)	$-(\forall x)(\exists y)Lxy$	2 QN (on a fragment of (2))

With QN, if you were permitted to operate on *wff*-fragments, *and* if you were permitted multiple such operations within a single *wff*:

1	(1)	$(\exists x)(\forall y)-Lxy$	A
1	(2)	$-(\forall x)(\exists y)Lxy$	QN (twice within (1))

3.S207: $(\forall x)(\exists y)-Lxy \vdash -(\exists x)(\forall y)Lxy$
Domain: People. 'Everyone is such that there's at least one person whom he (or she) doesn't love. Therefore no one loves everyone.'

Without QN (very difficult):

1	(1) $(\forall x)(\exists y)-Lxy$	A
2	(2) $(\exists x)(\forall y)Lxy$	A
3	(3) $(\forall y)Lxy$	A
3	(4) Lxy	3 ∀E
1	(5) $(\exists y)-Lxy$	1 ∀E
6	(6) $-Lxy$	A
3,6	(7) $Lxy \wedge -Lxy$	4,6 ∧E
6	(8) $-(\forall y)Lxy$	3,7 −I
	(9) $-Lxy \rightarrow -(\forall y)Lxy$	6,8 →I
1	(10) $-(\forall y)Lxy$	5,9 ∃E
1,3	(11) $(\forall y)Lxy \wedge -(\forall y)Lxy$	3,10 ∧I
3	(12) $-(\forall x)(\exists y)-Lxy$	1,11 −I
	(13) $(\forall y)Lxy \rightarrow -(\forall x)(\exists y)-Lxy$	3,12 →I
2	(14) $-(\forall x)(\exists y)-Lxy$	2,13 ∃E
1,2	(15) $(\forall x)(\exists y)-Lxy \wedge -(\forall x)(\exists y)-Lxy$	1,14 ∧I
1	(16) $-(\exists x)(\forall y)Lxy$	2,15 −I

3. S208: $-(\exists x)(\forall y)Lxy \vdash (\forall x)(\exists y)-Lxy$
Without QN: Difficult.
Domain: People. 'No one loves everyone. Therefore everyone is such that there's at least one person whom he (or she) doesn't love.'

4. S211: $(\exists x)(\forall y)Lxy \vdash (\forall y)(\exists x)Lxy$
Without QN:

1	(1) $(\exists x)(\forall y)Lxy$	A
2	(2) $(\forall y)Lxy$	A
2	(3) Lxy	2 ∀E
2	(4) $(\exists x)Lxy$	3 ∃I
2	(5) $(\forall y)(\exists x)Lxy$	4 ∀I
	(6) $(\forall y)Lxy \rightarrow (\forall y)(\exists x)Lxy$	2,5 →I
1	(7) $(\forall y)(\exists x)Lxy$	1,6 ∃E

Note that ∀I in (5) is legitimate because '*y*' in (4) doesn't appear free in (2), the *wff* on which (4) rests. Note too that you could have reversed the order of ∀I and ∃E.

4. S211: $(\exists x)(\forall y)Lxy \vdash (\forall y)(\exists x)Lxy$
With QN (although not required):

1	(1) $(\exists x)(\forall y)Lxy$	A
2	(2) $(\forall y)Lxy$	A
2	(3) Lxy	2 ∀E
4	(4) $-(\forall y)(\exists x)Lxy$	A
4	(5) $(\exists y)-(\exists x)Lxy$	4 QN
6	(6) $-(\exists x)Lxy$	A
6	(7) $(\forall x)-Lxy$	6 QN
6	(8) $-Lxy$	7 ∀E
2,6	(9) $Lxy \wedge -Lxy$	3.8 ∧I
6	(10) $-(\forall y)Lxy$	2,9 −I
	(11) $-(\exists x)Lxy \rightarrow -(\forall y)Lxy$	6,10 →I
4	(12) $-(\forall y)Lxy$	5,11 ∃E
2,4	(13) $(\forall y)Lxy \wedge -(\forall y)Lxy$	2,12 ∧I
2	(14) $--(\forall y)(\exists x)Lxy$	4,13 −I
2	(15) $(\forall y)(\exists x)Lxy$	14 −−E
	(16) $(\forall y)Lxy \rightarrow (\forall y)(\exists x)Lxy$	2,15 →I
1	(17) $(\forall y)(\exists x)Lxy$	1,16 ∃E

Moral of the story: Sometimes a shortcut isn't a shortcut.

4. S215: $(\exists x)(\forall y)(\exists z)Rxyz \vdash -(\forall x)(\exists y)(\forall z)-Rxyz$
Only a ten-line derivation—but *ridiculously difficult.*

Hints:
1. Spend a good bit of time—but not *all* your time—working from the bottom up.
2. Use QN several times—in the *counterintuitive* direction.

4. S216: $(\forall x)(\exists y)(\forall z)Rxyz \vdash -(\exists x)(\forall y)(\exists z)-Rxyz$
Without QN:

1	(1) $(\forall x)(\exists y)(\forall z)Rxyz$	A
2	(2) $(\exists x)(\forall y)(\exists z)-Rxyz$	A
3	(3) $(\forall y)(\exists z)-Rxyz$	A
3	(4) $(\exists z)-Rxyz$	3 ∀E
1	(5) $(\exists y)(\forall z)Rxyz$	1 ∀E
6	(6) $(\forall z)Rxyz$	A
6	(7) $Rxyz$	6 ∀E
8	(8) $-Rxyz$	A
6,8	(9) $Rxyz \wedge -Rxyz$	7,8 ∧I
8	(10) $-(\forall z)Rxyz$	6,9 —I
	(11) $-Rxyz \rightarrow -(\forall z)Rxyz$	8,10 →I
3	(12) $-(\forall z)Rxyz$	4,11 ∃E
3,6	(13) $(\forall z)Rxyz \wedge -(\forall z)Rxyz$	6,12 ∧I
6	(14) $-(\forall y)(\exists z)-Rxyz$	3,13 —I
	(15) $(\forall z)Rxyz \rightarrow -(\forall y)(\exists z)-Rxyz$	6,14 →I
1	(16) $-(\forall y)(\exists z)-Rxyz$	5,15 ∃E
1,3	(17) $(\forall y)(\exists z)-Rxyz \wedge -(\forall y)(\exists z)-Rxyz$	3,16 ∧I
3	(18) $-(\forall x)(\exists y)(\forall z)Rxyz$	1,17 —I
	(19) $(\forall y)(\exists z)-Rxyz \rightarrow -(\forall x)(\exists y)(\forall z)Rxyz$	3,18 →I
2	(20) $-(\forall x)(\exists y)(\forall z)Rxyz$	2,19 ∃E
1,2	(21) $(\forall x)(\exists y)(\forall z)Rxyz \wedge -(\forall x)(\exists y)(\forall z)Rxyz$	1,20 ∧I
1	(22) $-(\exists x)(\forall y)(\exists z)-Rxyz$	2,21 —I

4. S216: $(\forall x)(\exists y)(\forall z)Rxyz \vdash -(\exists x)(\forall y)(\exists z)-Rxyz$
With QN:

1	(1) $(\forall x)(\exists y)(\forall z)Rxyz$	A
1	(2) $(\exists y)(\forall z)Rxyz$	1 ∀E
3	(3) $(\forall z)Rxyz$	A
3	(4) $-(\exists z)-Rxyz$	3 QN
3	(5) $(\exists y)-(\exists z)-Rxyz$	4 ∃I
3	(6) $-(\forall y)(\exists z)-Rxyz$	5 QN
	(7) $(\forall z)Rxyz \rightarrow -(\forall y)(\exists z)-Rxyz$	3,6 →I
1	(8) $-(\forall y)(\exists z)-Rxyz$	2,7 ∃E
1	(9) $\forall x)-(\forall y)(\exists z)-Rxyz$	8 ∀I
1	(10) $-(\exists x)(\forall y)(\exists z)-Rxyz$	9 QN

The trick in using QN here is to adopt the Bottom-Up approach, using QN wherever possible, until you arrive at line (3), '$(\forall z)Rxyz$'. Then s witch to the Top-Down Approach and the game is over. Without QN: 22 lines; with QN: 10; enough said. Note that the ∀I that you performed on (8) you could *not* have performed on (6): (6) rests on (3) and x is free in (3).

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S231: $a = b, b \neq c \vdash a \neq c$

1	(1) $a = b$	A
2	(2) $b \neq c$	A
3	(3) $a = c$	A
1,3	(4) $b = c$	3,1 =E
1,2,3	(5) $(b = c) \wedge (b \neq c)$	4,2 ∧I
1,2	(6) $a \neq c$	3,5 —I

Two points:

1. If you find the move from (3) and (1) to (4) puzzling, think of (3) as ‘*Iac*’, i.e., as ‘*Pv*’, and think of (1) as ‘$v = t$’. ‘*v*’, of course, is the term to the left of the identity sign in ‘$v = t$’, and it’s also the term common to both ‘*Pv*’ and ‘$v = t$’. So ‘*v*’ in (3) and (1) must be ‘*a*’, and ‘*t*’ in (1) must be ‘*b*’. To generate (4), simply replace ‘*a*’ in (3) with ‘*b*’; i.e., simply replace ‘*v*’ in (3) with ‘*t*’.
2. No, the parentheses in (5) are *not* necessary.

S234: $(\exists x)(Px \wedge Lxa), a = b \vee a = c \vdash (\exists x)[Px \wedge (Lxb \vee Lxc)]$

1	(1) $(\exists x)(Px \wedge Lxa)$	A
2	(2) $a = b \vee a = c$	A
3	(3) $Px \wedge Lxa$	A
3	(4) Px	3 ∧E
3	(5) Lxa	3 ∧E
6	(6) $a = b$	A
3,6	(7) Lxb	5,6 =E
3,6	(8) $Lxb \vee Lxc$	7 ∨I
3,6	(9) $Px \wedge (Lxb \vee Lxc)$	4,8 ∧I
3	(10) $a = b \rightarrow [Px \wedge (Lxb \vee Lxc)]$	6,9 →I
11	(11) $a = c$	A
3,11	(12) Lxc	5,11 =E
3,11	(13) $Lxb \vee Lxc$	12 ∨I
3,11	(14) $Px \wedge (Lxb \vee Lxc)$	4,13 ∧I
3	(15) $a = c \rightarrow [Px \wedge (Lxb \vee Lxc)]$	11,14 →I
2,3	(16) $Px \wedge (Lxb \vee Lxc)$	2,10,15 ∨E
2	(17) $(Px \wedge Lxa) \rightarrow [Px \wedge (Lxb \vee Lxc)]$	3,16 →I
1,2	(18) $Px \wedge (Lxb \vee Lxc)$	1,17 ∃E
1,2	(20) $(\exists x)[Px \wedge (Lxb \vee Lxc)]$	18 ∃I

S237: Ta, $—(\exists x)(Tx \wedge x \neq a)$, $(\exists x)(\forall y)(Tx \leftrightarrow Hxy)$, $d \neq a \vdash (\exists z)(Hza \wedge —Hzd)$

1	(1) Ta	A
2	(2) $—(\exists x)(Tx \wedge x \neq a)$	A
3	(3) $(\exists x)(\forall y)(Ty \leftrightarrow Hxy)$	A
4	(4) $d \neq a$	A
5	(5) $(\forall y)(Ty \leftrightarrow Hxy)$	A
5	(6) $Ta \leftrightarrow Hxa$	5 ∀E
5	(7) $(Ta \rightarrow Hxa) \wedge (Hxa \rightarrow Ta)$	6 ↔E
5	(8) $Ta \rightarrow Hxa$	7 ∧E
1,5	(9) Hxa	8,1 →E
2	(10) $(\forall x)—(Tx \wedge x \neq a)$	2 QN
2	(11) $—(Td \wedge d \neq a)$	10 ∀E
2	(12) $—Td \vee —(d \neq a)$	11 DM
2	(13) $Td \rightarrow —(d \neq a)$	12 MI
14	(14) $—(d \neq a)$	A
4,14	(15) $(d \neq a) \wedge —(d \neq a)$	4,14 ∧I
4	(16) $——(d \neq a)$	14,15 —I
2,4	(17) $—Td$	13,16 MT
5	(18) $Td \leftrightarrow Hxd$	5 ∀E
5	(19) $(Td \rightarrow Hxd) \wedge (Hxd \rightarrow Td)$	18 ↔E
5	(20) $Hxd \rightarrow Td$	19 ∧E
2,4,5	(21) $—Hxd$	20,17 MT
1,2,4,5	(22) $Hxa \wedge —Hxd$	9,21 ∧I
1,2,4,5	(23) $(\exists z)(Hza \wedge —Hzd)$	22 ∃I
1,2,4	(24) $(\forall y)(Ty \leftrightarrow Hxy) \rightarrow (\exists z)(Hza \wedge —Hzd)$	5,23 →I
1,2,3,4	(25) $(\exists z)(Hza \wedge —Hzd)$	3,24 ∃E

Note the following:

1. In spite of the presence of the identity sign in (4), we never made use of either =E or =I.
2. We made use of ∀E twice on the same *wff*, (5), at lines (6) and (18), introducing a different term each time.
3. Sometimes, the most challenging part of a derivation is copying down correctly its premises and conclusion.

S239: $(\exists x)[Tx \land (\forall y)(Ty \rightarrow y = x)], Tb, b \neq c \vdash -Tc$

1	(1) $(\exists x)[Tx \land (\forall y)(Ty \rightarrow y = x)]$	A
2	(2) Tb	A
3	(3) $b \neq c$	A
4	(4) $Tx \land (\forall y)(Ty \rightarrow y = x)$	A
5	(5) Tc	A
4	(6) Tx	4 ∧E
4	(7) $(\forall y)(Ty \rightarrow y = x)$	4 ∧E
4	(8) $Tb \rightarrow b = x$	7 ∀E
2,4	(9) $b = x$	8,2 →E
4	(10) $Tc \rightarrow c = x$	7, ∀E
4,5	(11) $c = x$	10,5 →E
	(12) $c = c$	=I
4,5	(13) $x = c$	12,11 =E
2,4,5	(14) $b = c$	9,13 =E
2,3,4,5	(15) $b = c \land b \neq c$	14,3 ∧I
2,3,4	(16) $-Tc$	5,15 −I
2,3	(17) $[Tx \land (\forall y)(Ty \rightarrow y = x)] \rightarrow -Tc$	4,16 →I
1,2,3	(18) $-Tc$	1,17 ∃E

S244: $(\forall x)\{[Mx \wedge -(x = a \vee x = d)] \rightarrow Tx\}, Me \wedge -Te \vdash a = e \vee d = e$

1	(1) $(\forall x)\{[Mx \wedge -(x = a \vee x = d)] \rightarrow Tx\}$	A
2	(2) $Me \wedge -Te$	A
1	(3) $[Me \wedge -(e = a \vee e = d)] \rightarrow Te$	1 ∀E
4	(4) $-(a = e \vee d = e)$	A
4	(5) $-(a = e) \wedge -(d = e)$	4 DM
4	(6) $-(a = e)$	5 ∧E
4	(7) $-(d = e)$	5 ∧E
8	(8) $e = a$	A
	(9) $e = e$	=I
8	(10) $a = e$	9,8 =E
4,8	(11) $a = e \wedge -(a = e)$	10,6 ∧I
4	(12) $-(e = a)$	8,11 —I
13	(13) $e = d$	A
	(14) $e = e$	=I
13	(15) $d = e$	14,13 =E
4,13	(16) $d = e \wedge -(d = e)$	15,7 ∧I
8	(17) $-(e = d)$	13, 16 —I
4	(18) $-(e = a) \wedge -(e = d)$	2,17 ∧I
4	(19) $-(e = a \vee e = d)$	18 DM
2	(20) Me	2 ∧E
2,4	(21) $Me \wedge -(e = a \vee e = d)$	20, 19 ∧I
1,2,4	(22) Te	3,21 →E
1,2,4	(23) $-Te$	2 ∧E
1,2,4	(24) $Te \wedge -Te$	22,23 ∧I
1,2	(25) $--(a = e \vee d = e)$	4,24 —I
1,2	(26) $a = e \vee d = e$	25 ——E

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T42: $\vdash (\forall x)(Lx \vee -Lx)$

(1) $Lx \vee -Lx$	EM
(2) $(\forall x)(Lx \vee -Lx)$	1 ∀I

Line (1) is a perfectly legitimate springboard line for ∀I. As the substitution-instance of a previously proved theorem, (1) doesn't rest on any assumptions, and therefore *x* in (1) doesn't appear free in any assumptions upon which (1) rests.

T47: ⊢ (∀*y*)(∃*z*)[(∃*w*)(∀*x*)*Lwx* → *Lzy*]

I present two *attempts* at a proof of T47. They differ only in that one is correct and one isn't. Obvious question: Which is which? (The answer, and a brief explanation, follows.)

The first attempt:

1	(1) (∃*w*)(∀*x*)*Lwx*	A
2	(2) (∀*x*)*Lzx*	A
2	(3) *Lzy*	2 ∀E
	(4) (∀*x*)*Lzx* → *Lzy*	2,3 →I
1	(5) *Lzy*	1,4 ∃E
	(6) (∃*w*)(∀*x*)*Lwx* → *Lzy*	1,5 →I
	(7) (∃*z*)[(∃*w*)(∀*x*)*Lwx* → *Lzy*]	6 EI
	(8) (∀*y*)(∃*z*)[(∃*w*)(∀*x*)*Lwx* → *Lzy*]	7 ∀I

The second attempt at T47: $\vdash (\forall y)(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$:

1	(1)	$-(\forall y)(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	A
1	(2)	$(\exists y)-(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	1 QN
3	(3)	$-(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	A
3	(4)	$(\forall z)-[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	3 QN
3	(5)	$-[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	4 $\forall$E
3	(6)	$(\exists w)(\forall x)Lwx \wedge -Lzy$	5 NI
3	(7)	$(\exists w)(\forall x)Lwx$	6 $\wedge$E
3	(8)	$-Lzy$	6 $\wedge$E
9	(9)	$(\forall x)Lzx$	A
9	(10)	Lzy	9 $\forall$E
3,9	(11)	$Lzy \wedge -Lzy$	10,8 $\wedge$I
9	(12)	$--(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	3,11 $-$I
9	(13)	$(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	12 $--$E
	(14)	$(\forall x)Lzx \rightarrow (\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	9,13 $\rightarrow$I
3	(15)	$(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	7,14 $\exists$E
	(16)	$-(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy] \rightarrow (\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	3,15 $\rightarrow$I
1	(17)	$(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	2,16 $\exists$E
1	(18)	$(\forall y)(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	17 $\forall$I
1	(19)	$(\forall y)(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy] \wedge -(\forall y)(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	18,1 $\wedge$I
	(20)	$--(\forall y)(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	1,19 $-$I
	(21)	$(\forall y)(\exists z)[(\exists w)(\forall x)Lwx \rightarrow Lzy]$	20 $--$E

The second attempt succeeds; the first fails. The fatal mistake in the first occurs at line (5). ‘*z*’ is the variable that you introduced into your line-(2) ∃E assumption, and therefore it *must not* appear in line (5).